



UNIVERSITI PUTRA MALAYSIA

**PARALLEL BLOCK METHODS FOR SOLVING ORDINARY
DIFFERENTIAL EQUATIONS**

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FSAS 2004 20

**PARALLEL BLOCK METHODS FOR SOLVING
ORDINARY DIFFERENTIAL EQUATIONS**

By

ZANARIAH BT ABDUL MAJID

**Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in
Fulfilment of the Requirements for the Degree of Doctor of Philosophy**

June 2004



TO MY FAMILY

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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ORDINARY DIFFERENTIAL EQUATIONS**

By

ZANARIAH BT ABDUL MAJID

June 2004

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In this thesis, new and efficient codes are developed for solving Initial Value Problems (IVPs) of first and higher order Ordinary Differential Equations (ODEs) using variable step size. The new codes are based on the implicit multistep block methods formulae.

Subsequently, a more structured and efficient algorithm comprising the block methods was constructed for solving systems of first order ODEs using variable step size and order.

The new codes were then used for the parallel implementation in solving large systems of first and higher order ODEs. The sequential programs of these methods were executed on DYNIX/ptx operating system. The parallel programs were run on a Sequent Symmetry SE30 parallel computer.

The C^q stability in the multistep method was introduced and the focused was on the error propagation from a more practical angle.

The numerical results showed that the sequential implementation of the new codes could reduce the total number of steps and execution times even when solving small systems of first and higher order ODEs compared with the 1-point method and the existing 2PBVSO code in Omar (1999).

The parallel implementation of the codes was found to be most appropriate in solving large systems of first and higher order ODEs. It was also discovered that the maximum speed up of the parallel methods improved as the dimension of the ODEs systems increased.

In conclusion, the new codes developed in this thesis are suitable for solving systems of first and higher order ODEs.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

**KAEDAH BLOK SELARI BAGI MENYELESAIKAN
PERSAMAAN PEMBEZAAN BIASA**

Oleh

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Dalam tesis ini, suatu kod yang baru dan efisien dibentuk untuk menyelesaikan Masalah Nilai Awal (MNA) bagi sistem Persamaan Pembezaan Biasa (PPB) peringkat pertama dan tinggi menggunakan panjang langkah berubah. Kod baru ini berasaskan formula dari kaedah blok tersirat multilangkah.

Seterusnya, suatu algoritma dan kod yang lebih berstruktur dan efisien telah dibangunkan untuk menyelesaikan sistem PPB peringkat pertama menggunakan peringkat dan panjang langkah berubah.

Kod-kod baru ini dilanjutkan untuk pelaksanaan secara selari bagi sistem besar PPB peringkat pertama dan tinggi. Program jujukan dilaksana menggunakan sistem operasi DYNIX/ptx dan program selari menggunakan Sequent Symmetry SE30 iaitu komputer selari berkongsi ingatan.

Kestabilan C^q dalam kaedah multilangkah diperkenalkan dan tumpuan adalah pada penyebaran ralat dalam bentuk yang lebih praktikal.

Keputusan berangka menunjukkan pelaksanaan secara jujukan kod baru dapat mengurangkan langkah dan masa pelaksanaan penyelesaian walaupun melibatkan sistem kecil PPB peringkat pertama dan tinggi berbanding kaedah 1-titik dan kod 2PBVSO dalam Omar (1999).

Pelaksanaan secara selari kod baru tersebut amat sesuai bagi menyelesaikan sistem besar PPB peringkat pertama dan tinggi. Kelajuan maksimum kaedah blok selari dapat dicapai apabila dimensi sistem PPB meningkat.

Kesimpulannya, kod baru yang dibangunkan adalah sesuai bagi penyelesaian sistem PPB peringkat pertama dan tinggi.

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The Most Beneficent, The Most Merciful

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I certify that an Examination Committee has met on 22nd June 2004 to conduct the final examination of Zanariah bt. Abdul Majid on her Doctor of Philosophy thesis entitled "Parallel Block Methods For Solving Ordinary Differential Equations" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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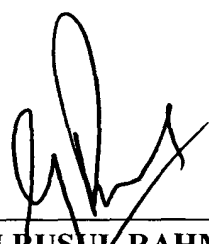
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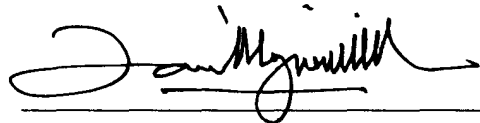
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DECLARATION

I hereby declare that the thesis is based on my original work, except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.



ZANARIAH BT ABDUL MAJID

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